

# Chern-Simons theory on an arbitrary manifold via surgery

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A general formula for physical observables in Chern-Simons theory with an arbitrary compact Lie group  $G$ , on an arbitrary closed oriented three-dimensional manifold  $\mathcal{M}$  is derived in terms of vacuum expectation values of Wilson loops in  $\mathcal{S}^3$ . Surgery presentation of  $\mathcal{M}$  and the Kirby moves are implemented as the main ingredients of the approach. The case of  $G = \text{SU}(n)$  is explicitly calculated.

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## 1. Introduction

Chern-Simons theory [1] is still a source of hopes and inspiration in both mathematics [2] (low-dimensional topology) and physics [3]. In this letter, we aim to *combinatorially* derive a general formula for physical observables (means of Wilson loops) in Chern-Simons theory with an arbitrary compact, simple, (gauge) Lie group  $G$ , on an arbitrary closed, connected, oriented three-dimensional manifold  $\mathcal{M}$ . The most non-trivial part of our task consists in proper encoding topological information about  $\mathcal{M}$  into the Chern-Simons path integral; the most convenient and elegant way to do it is seemingly the use of the *surgery prescription* [4]. Thus, our final formula will be expressed in terms of vacuum expectation values of Wilson loops in the three-dimensional sphere  $\mathcal{S}^3$ , where some of these Wilson loops will directly correspond to “physical observables”, whereas some of them (“non-physical” ones) will serve as a presentation of topology of  $\mathcal{M}$  in terms of surgery (compare with [5]). The “non-physical” Wilson loops will enter the theory through some specific linear combinations (“surgery loops”) being a subject of a new kind of (topological) symmetry. We will call this symmetry the *Kirby symmetry*, as it corresponds to the invariance of the surgery loops with respect to the *second Kirby move* [4]. From technical point of view, one should calculate the coefficients defining the surgery loops in terms of ordinary (“non-physical”) Wilson loops, taking into account the restrictions coming from the Kirby symmetry. It appears that the non-zero coefficients we are looking for are essentially uniquely given by the *quantum dimensions* of irreducible representations of  $G$ . The coefficients in the case of  $G = \text{SU}(n)$ ,  $n \geq 3$ , are explicitly calculated.

The proposed purely formal ideology can be supplemented with a more common interpretation in terms of particle physics. First of all, one should note that the surgery loop is nothing but an average of few Wilson loops corresponding to virtual “mesons” of different “isospins”, weighted by quantum dimensions. Thus, the change of topology

amounting to adding a two-handle is induced by a (particle) “virtual process”. Alternatively, the three-dimensional sphere  $\mathcal{S}^3$  in the presence of the virtual process is indistinguishable from the surgered sphere, at least as far as Chern-Simons interactions are concerned. This way, we obtain a very simple topology generating mechanism.

## 2. General formula

The vacuum expectation value of the observable  $\mathcal{O}$  in Chern-Simons theory with a compact (gauge) Lie group  $G$ , on  $\mathcal{S}^3$  is formally defined by the path integral [1]

$$\langle \mathcal{O} \rangle = \int \mathcal{O} e^{ikS_{\text{CS}}(A)} D\mathcal{A},$$

where

$$S_{\text{CS}}(A) = \frac{1}{4\pi} \int_{\mathcal{S}^3} \text{Tr} \left( AdA + \frac{2}{3} A^3 \right)$$

is the (properly normalized) Chern-Simons action functional of the gauge field  $A$ , the symbol  $D\mathcal{A}$  represents Feynman’s integral over all gauge orbits, and  $k \in \mathbf{Z}^+$  is the *level*. For instance, for Wilson loops in the fundamental representation of  $\text{SU}(2)$ , “ $\langle \rangle$ ” is the well-known *Kauffman bracket* (incidentally, in mathematical literature, denoted with the same symbol [6]); fundamental representations of  $\text{SU}(n)$ ,  $n \geq 3$ , correspond to *regular* (non-ambient) relatives of the *Homfly polynomial*. The standard definition of the Wilson loop round the *geometrical loop*  $\mathcal{C}$  reads

$$W_\mu = \text{Tr}_\mu \text{P exp} \oint_{\mathcal{C}} A,$$

where  $\mu$  numbers irreducible representations of  $G$ .

Now, we shall define the following “tensors”, needed for our further calculations:

$$d_\mu = \langle \bigcirc_\mu \rangle, \tag{1}$$

$$x_{\mu\nu} = \left\langle \cdots \bigodot_\mu \bigcup_\nu \cdots \right\rangle, \tag{2}$$

$$y_{\mu\nu\lambda} = y_{\nu\mu\lambda} = \left\langle \cdots \odot_{\mu\nu} \bigcup_{\lambda} \cdots \right\rangle, \quad (3)$$

where “o” denotes a trivial (untwisted) Wilson loop, “ $\odot$ ” a trivial but non-contractible Wilson loop in an annulus (a bordered exterior of “ $\bullet$ ”), in general, non-trivially immersed in  $\mathcal{S}^3$ , “ $\odot$ ” two parallel Wilson loops in the annulus, “ $\bigcup$ ” a piece of a Wilson loop, and finally “ $\cdots$ ” represents the rest that will be always identical on the both sides of any equation.

A particularly useful version of the *satellite formula* [7] is given by

$$y_{\mu\nu\lambda} = \sum_{\rho} c_{\rho\mu\nu} x_{\rho\lambda}, \quad (4)$$

where  $c_{\rho\mu\nu} = c_{\rho\nu\mu}$  are the *Clebsch-Gordan coefficients* corresponding to the decomposition of the representations inside the annulus

$$R_{\mu} \otimes R_{\nu} = \bigoplus_{\rho} c_{\rho\mu\nu} R_{\rho}.$$

In our notation, the Kirby symmetry assumes the following simple graphical form

$$\left\langle \cdots \odot_{\omega} \bigcup_{\Omega} \cdots \right\rangle = \left\langle \cdots \overset{\omega}{\bigcup}_{\Omega} \cdots \right\rangle, \quad (5)$$

where the loops denote now surgery loops rather than the Wilson ones, i.e.

$$W_{\omega}(A) = \sum_{\mu} \Delta_{\mu} W_{\mu}(A), \quad W_{\Omega}(A) = \sum_{\mu} \delta_{\mu} W_{\mu}(A), \quad (6a, b)$$

and  $\Delta_{\mu}$ ,  $\delta_{\mu}$  are the coefficients we are looking for. Applying the formula for a *connected sum* [7] to (3) yields

$$y_{\mu\nu\nu} = \left\langle \cdots \odot_{\mu\nu} \bigcup_{\nu} \cdots \right\rangle = d_{\nu} \left\langle \cdots \overset{\mu}{\bigcup}_{\nu} \cdots \right\rangle. \quad (7)$$

By virtue of (6), (7) and (4), we can rewrite RHS of (5) in the following manner

$$\left\langle \cdots \overset{\omega}{\bigcup}_{\Omega} \cdots \right\rangle = \sum_{\mu, \nu} \Delta_{\mu} \delta_{\nu} d_{\nu}^{-1} \sum_{\rho} c_{\rho\mu\nu} x_{\rho\nu}, \quad (8)$$

whereas LHS of (5) is of the form

$$\langle \cdots \odot_{\omega} \cup_{\Omega} \cdots \rangle = \sum_{\mu, \nu} \Delta_{\mu} \delta_{\nu} x_{\mu\nu}. \quad (9)$$

Since eq. (5) should be satisfied for any observables (and hence for any  $x_{\mu\nu}$ ), we have

$$\Delta_{\mu} \delta_{\nu} = \sum_{\rho} \Delta_{\rho} \delta_{\nu} d_{\nu}^{-1} c_{\mu\rho\nu}. \quad (10)$$

Finally (compare with ref. [8]),

$$d_{\nu} \Delta_{\mu} = \sum_{\rho} c_{\mu\rho\nu} \Delta_{\rho}, \quad (11)$$

where we have tacitly assumed that  $d_{\nu} \neq 0$  (in (8)), as well as  $\delta_{\nu} \neq 0$  (in (10)).

We should note that  $d_{\nu}$  can be identified to the quantum dimension of the representation  $\nu$  [9], and the “classical” *deformation parameter* expressed by the  $k^{\text{th}}$  primitive root of unity, where  $k$  is the level, undergoes a “quantum” shift [1] corresponding to

$$k \longrightarrow r = k + h, \quad (12a)$$

where  $h$  is the *dual Coxeter number*, i.e.

$$q_{\text{cl}} = e^{\frac{2\pi i}{k}} \longrightarrow q = e^{\frac{2\pi i}{r}}. \quad (12b)$$

The solution of eq. (11) in the form  $\Delta_{\nu} = d_{\nu}$  immediately follows from the locality of the theory, and from the satellite formula applied to a trivial Wilson loop, i.e.

$$\langle \bigcirc_{\nu} \rangle \langle \bigcirc_{\mu} \rangle = \langle \odot_{\nu\mu} \rangle = \sum_{\rho} c_{\rho\nu\mu} \langle \bigcirc_{\rho} \rangle, \quad (13a)$$

or by virtue of (1)

$$d_{\nu} d_{\mu} = \sum_{\rho} c_{\rho\nu\mu} d_{\rho}, \quad (13b)$$

as well as from the character formula for the quantum dimension and its reality ( $d_{\mu}^* = d_{\mu}$ ) for  $|q| = 1$  [10].

Since the roles of  $\omega$  and  $\Omega$  in (5) could be interchanged ( $\omega \leftrightarrow \Omega$ ) from the point of view of the Kirby symmetry, and we can build an arbitrary representation of the *primitive* ones, we should put  $\nu \in P$  (the set of primitive representations of  $G$ ) in (11) to assure the consistency. Then

$$\delta_\nu = \begin{cases} 1, & \text{if } \nu \in P, \\ 0, & \text{otherwise,} \end{cases}$$

and  $d_\nu \neq 0$  ( $\nu \in P$ ).

The observation that  $\nu$  should be restricted to primitive representations of  $G$  is crucial for two reasons: (i) eq. (11) becomes “irreducible”, i.e. there are no (and there should not be [11]) subsets of non-zero  $\Delta$ ’s not connected by (11), and hence the solution  $\Delta_\mu = d_\mu$  is essentially unique; (ii) eq. (11), looked at as a recursive relation, terminates yielding a finite set of representations.

Thus, physical observables in Chern-Simons theory with the (gauge) Lie group  $G$ , on the manifold  $\mathcal{M}$  are expressed by

$$\left\langle \prod_i W_{\mu_i}(A) \right\rangle_{\mathcal{M}} = \frac{\left\langle \prod_i W_{\mu_i}(A) \prod_j W_{\omega_j}(A) \right\rangle}{\left\langle \prod_j W_{\omega_j}(A) \right\rangle}, \quad (14)$$

where  $W_{\mu_i}(A)$  is a “physical” Wilson loop, and  $W_{\omega_j}(A)$  is

$$W_\omega(A) = \sum_{\mu} d_\mu W_\mu(A), \quad (15)$$

put on the  $j^{\text{th}}$  surgery loop entering the surgery presentation of  $\mathcal{M}$ .

### 3. $SU(n)$ example

According to [9], the quantum dimension of the representation corresponding to the highest weight  $\mu$  is

$$d_\mu = \prod_{\alpha \in \Phi^+} \frac{q^{(\mu+\rho, \alpha)/2} - q^{-(\mu+\rho, \alpha)/2}}{q^{(\rho, \alpha)/2} - q^{-(\rho, \alpha)/2}}, \quad (16)$$

where  $q$  is the deformation parameter (12b),  $\Phi^+$  is the set of the positive roots, and  $\rho$  is the *Weyl vector* (half the sum of the positive roots).

As an explicit example, we will consider the case of  $G = \text{SU}(n)$ ,  $n \geq 3$ . From eq. (16) it follows that

$$d_{\mu_1 \mu_2 \dots \mu_{n-1}} = \prod_{1 \leq i < j \leq n} \frac{q^{(\sum_{l=i}^{j-1} \mu_l + j - i)/2} - q^{-(\sum_{l=i}^{j-1} \mu_l + j - i)/2}}{q^{\frac{j-i}{2}} - q^{-\frac{j-i}{2}}}, \quad (17)$$

with  $\mu_l$  the Dynkin indices corresponding to  $\mu$ . It is obvious that  $d_{\mu_1 \mu_2 \dots \mu_{n-1}}$  vanishes the first time, truncating (11), for

$$\sum_{l=1}^{n-1} \mu_l = r + 1 - n = k + h + 1 - n = k + 1, \quad (18)$$

where eq. (12) as well as the fact that the dual Coxeter number  $h = n$  for  $G = \text{SU}(n)$  have been taken into account. Thus, only the representations corresponding to the Young diagrams contained in the rectangle with the sides of the length  $k$  and  $n - 1$  enter (15). Then, the recursive relation (11) terminates, and consequently

$$W_\omega(A) = \sum_{\substack{\mu_1, \mu_2, \dots, \mu_{n-1} \geq 0 \\ \sum_{l=1}^{n-1} \mu_l \leq k}} d_{\mu_1 \mu_2 \dots \mu_{n-1}} W_{\mu_1 \mu_2 \dots \mu_{n-1}}. \quad (19)$$

The expression (14) for  $G = \text{SU}(2)$ , modulo the normalization (corresponding to the first Kirby move), is known in literature as the invariant of Reshetikhin, Turaev and Witten.

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